Ordinary Differential Equations

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We have discussed antiderivatives of a given function $f(x)$, which can be obtained by computing the indefinite integral of $f(x)$, i.e., $F(x) = \int f(x)\,dx$. This is a good example of differential equations: recover the function $F(x)$ from some information about its derivatives or higher order derivatives.

More generally, an **ordinary differential equation** (ODE) is one of the form

$$G(x, y, y', ..., y^{(n)}) = 0$$

We assume $y = f(x)$ and solve for the expression of $f(x)$ from the above equation.

If the function involves more than one independent variables $x_1, ..., x_n$ instead of only an x, together with partial derivatives of y, and we solve for $y = f(x_1, ..., x_n)$, then the equation is called a **partial differential equation** (PDE).

ODE and PDE are two important subjects in modern mathematics.

We will learn to solve ODEs of the following form and also study some applications.

Definition 1. A **separable equation** is one of the form

$$y' = f(x)g(y)$$

Proposition 2. *A separable equation*

$$y' = f(x)g(y)$$

can be solved in the following steps:

1. *Rewrite the equation into the form*

$$\frac{1}{g(y)}\,dy = f(x)\,dx$$

2. Integrate both sides:

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

3. Obtain y in terms of x from the above equation.

Example 3. *The equation $y' = a + by$ can be solved by viewing it as a separable equation:*

$$\begin{aligned}
y' &= a + by \\
y' &= b(y + \frac{a}{b}) \\
\frac{1}{y + \frac{a}{b}}\,dy &= b\,dx \\
\int \frac{1}{y + \frac{a}{b}}\,dy &= \int b\,dx \\
\ln(y + \frac{a}{b}) &= bx + C \\
y + \frac{a}{b} &= e^{bx+C} \\
y &= -\frac{a}{b} + e^{bx}e^{C} \\
y &= Ae^{bx} - \frac{a}{b}
\end{aligned}$$

Example 4. *We can use this method to solve the differential equation*

$$\begin{cases} f'(x) = e^x f^2(x) \\ f(0) = 1 \end{cases}$$

We first write

$$\frac{1}{y^2}\,dy = e^x\,dx$$

Integrate both sides:

$$\int \frac{1}{y^2}\,dy = \int e^x\,dx + C$$

We get

$$-\frac{1}{y} = e^x + C$$

for some constant C. This implies

$$y = -\frac{1}{e^x + C}$$

We also know that $1 = f(0) = -\frac{1}{e^0+C}$, so $C = -2$. We conclude

$$y = -\frac{1}{e^x - 2}$$

Example 5. *Newtons Law of Cooling predicts the cooling of a warm body placed in a cold environment. According to the law, the rate at which the temperature of the body decreases is proportional to the difference of temperature between the body and its environment, i.e.*

$$\frac{dT}{dt} = k(T - T_e)$$

where T is the temperature of the object, T_e is the (constant) temperature of the environment, and k is a constant of proportionality. Given the initial temperature $T(0) = T_0$, we can recover the temperature function from this law:

$$\begin{aligned}\frac{dT}{dt} &= k(T - T_e) \\ \frac{1}{T - T_e}\,dT &= -k\,dt \\ \ln(T - T_e) &= -kt + C \\ T &= T_e + e^{-kt+C}\end{aligned}$$

When $t = 0, T = T_0$, so $T_0 = T_e + e^C$, we see $e^C = T_0 - T_e$, so

$$T = T_e + e^{-kt}(T_0 - T_e)$$

Observe that as $t \to +\infty$, $T \to T_e$, which means after long time, the temperature of the body will cool down to the environment temperature.

Example 6. *The Solow-Swan Model in economics is used to predict the long run growth in economics. It assumes at each time, the total production of* *some good (output) $Y(t)$ depends on the capital input $K(t)$ and labour input $L(t)$:*

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha}$$

where $0 < \alpha < 1$ is called the elasticity of the output with respect to capital input.

If We further assume the labour input is an exponential function of time given by

$$L(t) = e^{rt}$$

and the rate of increase of capital input is proportional to the total output at each time:

$$K'(t) = bY(t)$$

we can then solve for $K(t)$:

$$\begin{aligned}
\frac{1}{b}K' &= K^{\alpha}(e^{rt})^{1-\alpha} \\
\frac{1}{K^{\alpha}}\,dK &= be^{r(1-\alpha)t}\,dt \\
\int \frac{1}{K^{\alpha}}\,dK &= \int be^{r(1-\alpha)t}\,dt \\
\frac{1}{1-\alpha}K^{1-\alpha} &= \frac{b}{r(1-\alpha)}e^{r(1-\alpha)t} + C \\
K &= (\frac{b}{r}e^{r(1-\alpha)t} + C)^{\frac{1}{1-\alpha}}
\end{aligned}$$